RANDOM WALKS ON COUNTABLE GROUPS

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ABSTRACT. We begin by giving a new proof of the equivalence between the Liouville property and vanishing of the drift for symmetric random walks with finite first moments on finitely generated groups; a result which was first established by Kaimanovich-Vershik and Karlsson-Ledrappier. We then proceed to prove that the product of the Poisson boundary of any countable measured group (G,μ) with any ergodic $(G,\check{\mu})$-space is still ergodic, which in particular yields a new proof of weak mixing for the double Poisson boundary of (G,μ) when μ is symmetric. Finally, we characterize the failure of weak-mixing for an ergodic (G,μ)-space as the existence of a non-trivial measure-preserving isometric factor.

1. MEASURED GROUPS AND THEIR POISSON BOUNDARIES

A *measured group* is a pair (G,μ), where G is a countable group and μ is a probability measure on G whose support generates G as a semi-group. We say that μ is *symmetric* if the *adjoint* probability measure $\check{\mu}(s) := \mu(s^{-1})$ coincides with μ. We write $\mu^0 = \delta_e$ and

$$\mu^{*n}(s) = \sum \mu(s_1)\cdots\mu(s_n), \quad \text{for } s \in G \text{ and } n \geq 1,$$

where the sum is taken over all n-tuples $(s_1,\dots,s_n)$ in G^n such that $s_1\cdots s_n = s$. A real-valued function u on G is *μ-harmonic* if

$$\sum_s u(gs)\mu(s) = u(g), \quad \text{for all } g \in G.$$

Let $\ell^\infty(G)$ denote the Banach space of real-valued *bounded* functions on G endowed with the supremum norm, and let $H^\infty(G,\mu)$ denote the closed sub-space of $\ell^\infty(G)$ consisting of bounded μ-harmonic functions. We say that (G,μ) is *Liouville* if $H^\infty(G,\mu)$ consists only of constant functions.

Let (X,ν) be a Borel probability measure space, and denote by $L^\infty(X,\nu)$ the Banach space of ν-essentially bounded real-valued functions on X, identified up to ν-null sets, endowed with the (essential) supremum norm. Suppose that X is equipped with an action of G by measurable maps, which preserve the class of ν-null sets in X, and whose inverses are also measurable. We say the action is *ergodic* if whenever $B \subset X$ is a Borel set such that $\nu(B\Delta g\cdot B) = 0$ for all $g \in G$, where Δ denotes the symmetric difference of sets, then B is either a ν-null set or a ν-conull set. We say that ν is *μ-stationary*, and that (X,ν) is a *(G,μ)-space*, if

$$\sum_{s\in G}\Big(\int_X f(sx)\,d\nu(x)\Big)\mu(s) = \int_X f(x)\,d\nu(x), \quad \text{for all } f \in L^\infty(X,\nu).$$

We note that if (X,ν) is a (G,u)-space and $f \in L^\infty(X,\nu)$, then the image of the bounded linear map $P_\nu : L^\infty(X,\nu) \to \ell^\infty(G)$ defined by

$$P_\nu f(s) := \int_X f(sx)\,d\nu(x), \quad \text{for } s \in G \text{ and } f \in L^\infty(X,\nu),$$

is contained in $H^\infty(G,\mu)$.

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Theorem 1.1 (Furstenberg [2]). *For every measured group (G,μ) there exists an ergodic (G,μ)-space (Z,m) such that the bounded linear map $P_m : L^\infty(Z,m) \to H^\infty(G,\mu)$ defined above is an isometric isomorphism of Banach spaces.*

In particular, (G,μ) is Liouville if and only if (Z,m) is trivial, i.e. if the support of m consists of one point, which happens if and only if m is G-invariant.

Remark 1.1. Furthermore, up to G-equivariant measurable isomorphisms, (Z,m) is uniquely determined, and we shall refer to any representative of (Z,m) as the *Poisson boundary* of (G,μ).

Let $L^1(Z,m)$ denote the Banach space of ν-integrable functions on Z, identified up to null sets, endowed with the L^1-norm. Using Hahn-Banach's Theorem and the fact that $L^1(Z,m)^* \cong L^\infty(Z,m)$, we get the following reformulation of Furstenberg's Theorem:

$$\overline{\operatorname{span}}\Big\{\frac{ds_*m}{dm} : s \in G\Big\} = L^1(Z,m). \tag{1.1}$$

2. Zero drift vs. Liouville

Suppose that ρ is a *semi-norm* on G, i.e. ρ is a non-negative function on G such that

$$\rho(e) = 0 \quad \text{and} \quad \rho(s) = \rho(s^{-1}) \quad \text{and} \quad \rho(st) \le \rho(s) + \rho(t), \quad \text{for all } s,t \in G. \tag{2.1}$$

If, in addition, $\rho(s) = 0$ implies that $s = e$, then we say that ρ is a *norm* on G. For instance, if G is finitely generated and $S \subset G$ is a finite generating set with $S^{-1} = S$, then

$$\rho_S(s) = \inf\{n \ge 1 : s \in S^n\}, \quad \text{for } s \in G,$$

is a norm, often referred to as the *word-norm* associated to S. Given a semi-norm ρ on G, we define the *drift* $\ell_\rho(\mu)$ of the triple (G,μ,ρ) by

$$\ell_\rho(\mu) = \lim_n \frac{1}{n} \sum_s \rho(s)\mu^{*n}(s).$$

The limit exists by sub-additivity and is finite if ρ is μ-integrable. The aim of this section is to give a new proof of the following theorem:

Theorem 2.1. *Let (G,μ) be a finitely generated symmetric measured group and let ρ be a word-norm. Suppose that ρ is μ-integrable. Then (G,μ) is Liouville if and only if $\ell_\rho(\mu) = 0$.*

Remark 2.1. The direction "Zero drift implies Liouville" was proved by Kaimanovich-Vershik in [5] using the Avez entropy of random walks, and the direction "Liouville implies Zero drift" was proved by Karlsson-Ledrappier in [7] using their Multiplicative Ergodic Theorem. An alternative proof was later given by Erschler-Karlsson in [1].

2.1. **The Poisson semi-norm.** Let (Z,m) be the Poisson boundary of (G,μ) and define the (multiplicative) *Poisson cocycle* $\sigma(s,z) = \frac{ds_*m}{dm}(z)$, which is well-defined on a G-invariant m-conull subset $Z' \subset Z$. One readily verifies the relations

$$\sigma(st,z) = \sigma(s,z)\sigma(t,s^{-1}z), \quad \text{for all } s,t \in G \text{ and } m\text{-a.e. } z, \tag{2.2}$$

and

$$\sum_{s\in G} \sigma(s,z)\mu^{*k}(s) = 1, \quad \text{for all } k \ge 1 \text{ and } m\text{-a.e. } z.$$

These relations in particular imply that

$$\rho_\mu(s) = \log\|\sigma(s,\cdot)\|_\infty, \quad \text{for } s \in G,$$

defines a semi-norm on G, which we shall refer to as the *Poisson semi-norm* of (G,μ).

2.2. **Zero drift implies Liouville.** Let (G,μ) be a finitely generated measured group and let ρ be a word-norm on G. Suppose that ρ is μ-integrable and satisfies $\ell_\rho(\mu)=0$. There exists a constant C_μ such that $\rho_\mu \le C_\mu \rho$, where ρ_μ is the Poisson semi-norm defined above, so in particular, we conclude that ρ_μ is μ-integrable and $\ell_{\rho_\mu}(\mu)=0$.

Let (Z,m) denote the Poisson boundary of (G,μ) and define the sequence

$$c_n := \sum_s \Big(\int_Z \log \sigma(s,z)\, dm(z) \Big)\, d\check{\mu}^{*n}(s), \quad \text{for } n \ge 1,$$

where σ denotes the Poisson cocycle defined above. One readily verifies that $c_{n+m} = c_n + c_m$ for all $m,n \ge 1$, and thus

$$\begin{aligned} 0 &= \ell_{\rho_\mu}(\mu) = \ell_{\rho_\mu}(\check{\mu}) = \lim_n \frac{1}{n} \sum_s \log \|\sigma(s,\cdot)\|_\infty \check{\mu}^{*n}(s) \ge \liminf_n -\frac{1}{n} \sum_s \Big(\int_Z \log \sigma(s,z)\, dm(z) \Big) \check{\mu}^{*n}(s) \\ &= \liminf_n -\frac{c_n}{n} = -c_1 = -\sum_s \Big(\int_Z \log \sigma(s,z)\, dm(z) \Big) \check{\mu}(s) \ge -\sum_s \log \Big(\int_Z \sigma(s,z)\, dm(z) \Big) \check{\mu}(s) = 0, \end{aligned}$$

where we in the second to last step used Jensen's inequality. We conclude that

$$\sum_s \Big(\int_Z \log \sigma(s,z)\, dm(z) \Big) \check{\mu}(s) = 0,$$

and thus $\sigma(s,z)=1$ m-almost everywhere, for all $s \in \operatorname{supp} \check{\mu}$. By (2.2) we conclude that $\sigma(s,\cdot)=1$ m-almost everywhere for all $s \in G$, and thus m is G-invariant.

We stress that we did not use the assumption that μ is symmetric in this proof. In particular, since $\ell_\mu(\rho) = \ell_\rho(\check{\mu})$, we have the following corollary.

Corollary 2.2. *Let (G,μ) be a finitely generated measured group and let ρ be a word-norm on G. Assume that ρ is μ-integrable. Then (G,μ) is Liouville if and only if $(G,\check{\mu})$ is Liouville.*

There are examples (see e.g. [5]) of finitely generated measured groups which are Liouville, while their adjoint measured groups are not.

2.3. **Liouville implies zero drift.** A real-valued function ϕ on G is called *left Lipschitz* if

$$\sup_s \big|\phi(gs) - \phi(s)\big| < \infty, \quad \text{for all } g \in G,$$

and *quasi-μ-harmonic with distortion ℓ*, where ℓ is a real number, if ϕ is μ-integrable and satisfy

$$\sum_s \phi(gs)\,\mu(s) = \phi(g) + \ell, \quad \text{for all } g \in G.$$

In particular, if $\ell = 0$, then ϕ is μ-harmonic. The remaining direction in Theorem 2.1 is now readily deduced from the following two lemmata.

Lemma 2.3. *Let ρ be a norm on G which is μ-integrable. Then there exists a left Lipschitz quasi-μ-harmonic function on G with $\phi(e)=0$ and distortion $\ell_\rho(\mu)$.*

Lemma 2.4. *Suppose that (G,μ) is Liouville. If ϕ is a left Lipschitz quasi-μ-harmonic function on G with distortion ℓ and $\phi(e)=0$, then ϕ is a homomorphism from G into $\mathbb{R}$. In particular, if μ is symmetric, then $\ell = 0$.*

Proof of Lemma 2.3. By telescoping, we have

$$\ell_\rho(\mu) = \lim_n \frac{1}{n} \sum_{k=0}^{n-1} \sum_{s,t} \big(\rho(st) - \rho(t)\big)\,\mu^{*k}(t)\,\mu(s). \tag{2.3}$$

Define the functions

$$f_k(s) := \sum_t \big(\rho(st) - \rho(t)\big)\,\mu^{*k}(t), \quad \text{for } s \in G,$$

and note that by the triangle inequality, $|f_k(s)| \leq \rho(s)$ for all s. Since ρ is assumed to be μ-integrable, the sequence

$$\phi_n = \frac{1}{n}\sum_{k=0}^{n-1} f_k(s), \quad \text{for } s \in G,$$

is μ-dominated, and by a simple diagonal argument, we can find a sub-sequence (n_j) such that the function

$$\phi(s) := \lim_j \phi_{n_j}(s) \quad \text{exists for all } s \in G.$$

In particular, by dominated convergence, we have

$$\phi(e) = 0 \quad \text{and} \quad \sum_s \phi(s)\,\mu(s) = \ell_\mu(\rho).$$

It remains to prove that ϕ is quasi-μ-harmonic and left Lipschitz. Since we can write

$$\rho(gst) - \rho(t) = \rho(gst) - \rho(st) + \rho(st) - \rho(t), \quad \text{for all } g, s, t \in G,$$

we have $|\phi_n(gs) - \phi_n(s)| \leq \rho(g)$ for all s and n, and thus ϕ is left Lipschitz. Furthermore,

$$\sum_s f_k(gs)\,\mu(s) = f_{k+1}(g) + \sum_s f_k(s)\,\mu(s) \tag{2.4}$$

for all k. Hence,

$$\sum_s \phi(gs)\,\mu(s) = \lim_j \frac{1}{n_j}\sum_{k=0}^{n_j-1}\sum_s f_k(gs)\,\mu(s) = \lim_j \frac{1}{n_j}\sum_{k=0}^{n_j-1} f_{k+1}(g) + \lim_j \frac{1}{n_j}\sum_{k=0}^{n_j-1}\sum_s f_k(s)\mu(s),$$

which clearly converges to $\phi(g) + \ell_\rho(\mu)$ for all $g \in G$. □

Proof of Lemma 2.4. Fix $g \in G$ and define $u(s) = \phi(gs) - \phi(s)$. Since ϕ is left Lipschitz and quasi-μ-harmonic, we readily see that u is a bounded μ-harmonic function, and thus constant since (G,μ) is Liouville. By evaluating u at e and using $\phi(e) = 0$, we conclude that $\phi(gs) - \phi(s) = \phi(g)$ for all $g, s \in G$ and thus ϕ is a homomorphism. □

3. Ergodicity of products

The main aim of this section is to prove the following theorem.

Theorem 3.1. *Let (Z,m) denote the Poisson boundary of (G,μ) and let $(X,\check{\nu})$ be an ergodic $(G,\check{\mu})$-space. Then the diagonal action $G \curvearrowright (Z \times X, m \otimes \check{\nu})$ is ergodic.*

As an immediate consequence of this theorem, we deduce an important special case of a result by Kaimanovich [6]:

Corollary 3.2. *Let (Z,m) and $(\check{Z},\check{m})$ denote the Poisson boundaries of (G,μ) and $(G,\check{\mu})$ respectively. Let (Y,η) be an ergodic probability measure-preserving G-space. Then the diagonal action $G \curvearrowright (Z \times \check{Z} \times Y, m \otimes \check{m} \otimes \eta)$ is ergodic.*

Proof. We shall use Theorem 3.1 twice. First, note that since η is preserved by the G-action, the product space $(\check{Z} \times Y, \check{m} \times \eta)$ is an ergodic $(G,\check{\mu})$-space by Theorem 3.1. Hence, by Theorem 3.1, the diagonal G-action on the product of (Z,m) and $(\check{Z} \times Y, \check{m} \times \eta)$ is ergodic. □

We shall prove that if $F \in L^\infty(Z \times X, m \otimes \check{\nu})$ is essentially G-invariant and $\int_{Z\times X} F\,dm \otimes \check{\nu} = 0$, then $F = 0$ almost everywhere, or equivalently, for every $\phi \in L^1(Z,m)$,

$$\int_Z \phi(z)F(z,x)\,dm(z) = 0, \quad \text{for } \check{\nu}\text{-a.e. } x \in X.$$

Since the linear span of all functions of the form $\frac{ds_*m}{dm}$, where s ranges over G, is norm dense in the Banach space $L^1(Z,m)$ by (1.1), we are left with proving

$$\int_Z \frac{ds_*m}{dm}(z)F(z,x)\,dm(z) = \int_Z F(sz,x)\,dm(z) = \int_Z F(z,s^{-1}x)\,dm(z) = 0$$

for $\check{\nu}$-a.e. $x \in X$ and for all $s \in G$. Let $f(x) = \int_Z F(z,x)\,dm(z)$ and note that $f \in L^\infty(X,\check{\nu})$ and

$$\sum_s f(sx)\check{\mu}(s) = \sum_s \Big(\int_Z F(sz,x)\,dm(z)\Big)\,d\mu(s) = f(x),$$

since m is μ-stationary. We wish to prove that f vanishes $\check{\nu}$-almost everywhere, or, what amounts to the same, that f is $\check{\nu}$-essentially constant. Indeed, if f is essentially constant, then

$$f(x) = \int_Z F(z,x)\,dm(z) = \int_{Z\times X} F\,dm\otimes\check{\nu} = 0, \quad \text{for } \check{\nu}\text{-a.e } x \in X,$$

and thus the following lemma, applied to the ergodic $(G,\check{\mu})$-space $(X,\check{\nu})$, finishes the proof of Theorem 3.1.

Lemma 3.3. *Let (X,ν) be a (G,μ)-space. If $f \in L^\infty(X,\nu)$ satisfies*

$$\sum_s f(sx)\,\mu(s) = f(x), \quad \textit{for } \nu\textit{-a.e. } x \in X,$$

then f is G-invariant. In particular, if (X,ν) is ergodic, then f is essentially constant.

Proof. Since the support of μ generates G as a semi-group, it suffices to show that

$$\sum_s \Big(\int_X \big|f(sx)-f(x)\big|^2\,d\nu(x)\Big)\,\mu^{*k}(s) = 0$$

for all k. Upon expanding the square and using the fact that (X,ν) is a (G,μ)-space, we note that

$$\sum_s \Big(\int_X (f(sx)-f(x))^2\,d\nu(x)\Big)\,\mu^{*k}(s) = 2\Big(\int_X f^2\,d\nu - \int_X f(x)\Big(\sum_s f(sx)\,\mu^{*k}(s)\Big)\,d\nu(x)\Big).$$

By our assumption on f, we conclude that these expressions vanish for every k. □

4. Failure of weak-mixing

We recall that a *factor* of a G-space (X,ν) is a G-space (W,ξ) together with a G-equivariant Borel map $p : X' \to W$, where $X' \subset X$ is a G-invariant ν-conull set such that

$$p_*\nu(A) = \nu(p^{-1}(A)) = \xi(A), \quad \text{for all Borel sets } A \subset W.$$

We say that the G-spaces (X,ν) and (W,ξ) are *isomorphic* if p admits a measurable inverse map $q : W' \to X$, defined on a G-invariant subset $W' \subset W$, such that $q_*\xi = \nu$.

Let K be a compact and second countable group and suppose that $\tau : G \to K$ is a homomorphism with dense image. Given a closed subgroup $L < K$, we denote by $m_{K/L}$ the Haar probability measure on K/L, and we note that $(K/L, m_{K/L})$ is a G-space under the G-action $gkL = \tau(g)kL$, for the coset $kL \in K/L$. We say that a G-space (W,ξ) is *isometric* if it is isomorphic to a G-space of the form $(K/L, m_{K/L})$ for some compact and second countable group K, closed subgroup L of K and a homomorphism $\tau : G \to K$ with dense image. Since we assume that our compact groups are second countable, the associated L^2-spaces (with respect to the Haar probability measures) are separable in the weak topology.

The main aim of this section is to prove the following theorem.

Theorem 4.1. *Let (X,ν) be an ergodic (G,μ)-space and (Y,η) an ergodic probability measure-preserving G-space. Suppose that the diagonal action $G \curvearrowright (X \times Y, \nu \otimes \eta)$ is* not *ergodic. Then there exists a non-trivial factor of (X,ν) which is an isometric G-space.*

Remark 4.1. In a remark in an earlier pre-print of this paper, we discussed an alternative approach to this theorem, based on some dynamical properties of the WAP-compactification of G established earlier by Furstenberg-Glasner in [3]. Recently, Glasner-Weiss expanded on this remark, and gave a full alternative proof of Theorem 4.1 in [4].

We need some notation. Let B_1 denote the unit ball in the *real* Hilbert space $L^2_o(Y,\eta)$ of real-valued square-integrable functions with zero integrals, identified up to null sets, endowed with the weak topology, which makes B_1 into a compact and second countable space. The regular (Koopman) representation π on $L^2_o(Y,\eta)$ defined by $\pi(s)u(y) = u(s^{-1}y)$ is unitary, and gives rise to a weakly continuous action on B_1 via $su = \pi(s)u$ for $s \in G$ and $u \in B_1$. Theorem 4.1 is an immediate consequence of the following two lemmata.

Lemma 4.2. *Suppose that $f \in L^\infty(X \times Y, \nu \otimes \eta)$ is essentially G-invariant and has zero $\nu \otimes \eta$-integral, but does not vanish almost everywhere. If $\|f\|_\infty \leq 1$, then $p_f : X \to B_1$ given by $p_f(x) = f(x,\cdot)$ is a factor map, where $\nu_f = (p_f)_*\nu$. Furthermore, the Borel function*

$$\Phi(u,v) = \langle u,v\rangle_{L^2_o(Y,\eta)}, \quad \textit{for } u,v \in B_1,$$

is not *$\nu_f \otimes \nu_f$-essentially constant on $B_1 \times B_1$. In particular, (B_1,ν_f) is a non-trivial ergodic (G,μ)-space.*

Remark 4.2. To see why $p_f(x) \in B_1$, note that $\|p_f(x)\|_{L^2} \leq \|p_f(x)\|_\infty \leq 1$ and the function

$$h(x) = \int_Y p_f(x)\,d\nu = \int_Y f(x,y)\,d\eta(y)$$

is essentially G-invariant, and thus essentially constant by ergodicity of (X,ν). Since f is assumed to have $\nu \otimes \eta$-integral equal to zero, we conclude that h vanishes almost everywhere, and thus $p_f(x) \in L^2_o(Y,\eta)$ for ν-a.e. x in X.

In the next lemma, $\mathscr{H}$ denotes a real (separable) Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and B_1 denotes the unit ball in $\mathscr{H}$ endowed with the weak topology. We assume that π is a unitary representation of G on $\mathscr{H}$, and consider B_1 as a G-space under the action $gu = \pi(g)u$.

Lemma 4.3. *Suppose that ξ is a μ-stationary Borel probability measure on B_1. Then ξ is G-invariant. Furthermore, if the Borel map $(u,v) \mapsto \langle u,v\rangle$ is* not *$\xi\otimes\xi$-essentially constant on $B_1\times B_1$, then (B_1,ξ) admits a non-trivial factor which is isometric.*

Proof of Lemma 4.2. The assertion that p_f defines a factor map into (B_1,ν_f) is easy, so what remains to prove is that Φ is not $\nu_f \otimes \nu_f$-almost everywhere equal to a constant c, or, what amounts to the same thing, the function f_2 on $X \times X$ defined by

$$f_2(x,z) := \Phi(p_f(x),p_f(z)) = \langle p_f(x),p_f(z)\rangle_{L^2_o(Y,\eta)} = \int_Y f(x,y)f(z,y)\,d\eta(y)$$

is not $\nu \otimes \nu$-almost everywhere equal to c. We first claim that if $f_2 = c$ almost everywhere, then $c = 0$. Indeed, note that

$$c = \int_X\int_X f_2(x,z)\,d\nu(x)d\nu(z) = \int_Y \Big(\int_X f(x,y)\,d\nu(x)\Big)^2 d\eta(y). \tag{4.1}$$

Since f is G-invariant and (X,ν) is a (G,μ)-space, the function

$$\Lambda(y) = \int_X f(x,y)\,d\nu(x)$$

satisfies

$$\sum_{s\in G}\Lambda(sy)\check{\mu}(s)=\sum_{s\in G}\Big(\int_X f(x,s^{-1}y)\,d\nu(x)\Big)\mu(s)=\sum_{s\in G}\Big(\int_X f(sx,y)\,d\nu(x)\Big)\mu(s)=\Lambda(y).$$

By Lemma 3.3, applied to Λ and the $(G,\check{\mu})$-space (Y,η) - note that every probability measure preserving G-space is automatically a (G,p)-space for any probability measure p on G - we conclude that Λ is G-invariant and thus essentially constant by ergodicity of (Y,η). Since

$$\int_Y \Lambda(y)\,d\eta(y)=\int_X\int_Y f(x,y)\,d\nu(x)\,d\eta(y)=0,$$

by our assumption on f, we see that Λ vanishes almost everywhere. From (4.1), we conclude that $c=0$. Hence it suffices to prove that f_2 does not vanish $\nu\otimes\nu$-almost everywhere. Assume that $f_2(x,z)=0$ almost everywhere, so that

$$\int_{X\times X} f_2(x,z)\psi(x)\psi(z)\,d\nu(x)\,d\nu(z)=\int_Y\Big|\int_X f(x,y)\psi(x)\,d\nu(x)\Big|^2 d\eta(y)=0,$$

for all $\psi\in L^\infty(X,\nu)$. Then,

$$\int_X f(x,y)\psi(x)\,d\nu(x)=0,\quad \text{for } \eta\text{-a.e. } y \text{ and for all } \psi\in L^\infty(X,\nu),$$

which readily implies that f vanishes identically, and this contradiction finishes the proof. $\square$

Proof of Lemma 4.3. We first prove that ξ is G-invariant. Let $C(B_1)$ denote the real Banach space of continuous functions on B_1, equipped with the uniform norm and let $\mathscr{F}\subset C(B_1)$ be the sub-algebra of $C(B_1)$ generated by the constant functions and all functions on the form

$$\phi(v)=\langle u_1,v\rangle\cdots\langle u_k,v\rangle,\quad \text{for } v\in B_1,$$

where $u_1,\dots,u_k$ range over all finite lists of vectors in B_1. Since $\mathscr{F}$ separates points and does not vanish anywhere, we conclude by Stone-Weierstrass Theorem, that $\mathscr{F}$ is uniformly dense in $C(B_1)$. For a fixed list $u_1,\dots,u_k$ and with ϕ as above, we note that

$$\int_{B_1}\phi(sv)\,d\xi(v)=\int_{B_1}\langle u_1,\pi(s)v\rangle\cdots\langle u_k,\pi(s)v\rangle\,d\xi(v)=\langle u_1\otimes\cdots\otimes u_k,\pi^{\otimes k}(s)\sigma_k\rangle_{\mathscr{H}^{\otimes k}},$$

where $\sigma_k=\int_{B_1} v\otimes\cdots\otimes v\,d\xi(v)$. Since ξ is μ-stationary and $u_1,\dots,u_k$ are arbitrary, we conclude that

$$\sum_{s\in G}\mu(s)\pi^{\otimes k}(s)\sigma_k=\sigma_k.$$

Since norm balls in $\mathscr{H}^{\otimes k}$ are strictly convex, we see that σ_k is fixed by all $\pi^{\otimes k}(s)$ when s ranges over $\operatorname{supp}\mu$. Since the support of μ is assumed to generate G, we conclude that σ_k is fixed by G, and thus

$$\int_{B_1}\phi(sv)\,d\xi(v)=\int_{B_1}\phi(v)\,d\xi(v),\quad \text{for all } s\in G \text{ and } \phi\in\mathscr{F}.$$

Since $\mathscr{F}$ is uniformly dense in $C(B_1)$, we see that ξ is G-invariant.

Since $\Phi(u,v)=\langle u,v\rangle$ is a G-invariant Borel function on $B_1\times B_1$ which is not essentially constant with respect to $\xi\otimes\xi$, there exists by Mackey's Theorem [8], a non-trivial isometric factor of (B_1,ξ). $\square$

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